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ARTICLES

Thermodynamic renormalization group

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For a system with a first or second order phase transition at T^*, we construct a type of renormalization group transformation that maps the reduced temperature $t=(T-T^*)/T^*$ and the external field h for a system of volume V to t' and h' for another system of volume V'. Our construction is purely thermodynamic without reference to effective Hamiltonians. Using the transformation, we derive the finite size scaling form for the singular part of the free energy, which leads to the scaling laws in the thermodynamic limit. We thus provide a unified thermodynamic framework for phase transitions.

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I. INTRODUCTION

A phase transition, whether it is continuous (of second order) or discontinuous (of first order), is controlled by a singularity in the free energy density at the thermodynamic limit, where the system volume becomes infinite while its density is kept constant. For a finite system, however, the free energy density is normally an analytic function of thermodynamic variables and therefore exhibits no singularity. The question is then how the singular function f_∞^S or the singular part of the free energy density emerges as the $V=\infty$ limit for a sequence of the analytic functions $\{f_V^S\}$, where f_V^S is a part of the free energy density for a finite system with volume V. The answer was given by the finite size scaling ansatz postulated by Fisher and by Fisher and Barber [1]. This ansatz was later derived through the renormalization group (RG) approach [2], which has thus far provided a systematic understanding of phase transitions in terms of RG transformations among effective Hamiltonians or coarse-grained effective free energies defined on different length scales. However, the finite size scaling ansatz has never been derived within thermodynamics (i.e., without referring to effective Hamiltonians).

In this paper we will derive, within thermodynamics, the finite size scaling form for the singular part of the free energy of a system with a first or second order phase transition. We will achieve this goal by constructing, on the macroscopic level, a type of RG transformation among systems of different volumes. We will start from a minimal set of basic assumptions on the singular parts of the free energy and of the free energy density both near and at the thermodynamic limit. Our derivation will be purely thermodynamic: we will neither refer to effective Hamiltonians nor use conventional RG transformations among them. To distinguish our RG transformations from the conventional ones, we will call ours *thermodynamic* RG transformations (TRGT's).

For second order transitions, the finite size scaling form that we will derive leads directly to Widom and Kadanoff's scaling hypotheses at the thermodynamic limit, and consequently to the scaling laws among the critical exponents. For first order transitions, our finite size scaling form is the one postulated by Fisher and Berker [3]. The TRGT will thus provide a unified thermodynamic framework for both first and second order phase transitions.

For simplicity, we will assume our system to be a d-dimensional ferromagnetic system in a d-dimensional hypercube of volume V. The linear dimension L of the system is then given by $V=L^d$, and we take the thermodynamic limit for the systems by letting V go to infinity while we keep the density of the system fixed. We will assume that our system is in thermal contact with a heat reservoir at temperature T, and is also subject to an external magnetic field h. All the thermodynamic quantities of our system are then functions of three variables: the reduced temperature $t\equiv(T-T^*)/T^*$, where T^* is the phase transition temperature, the external field h, and the system volume V. The phase transition point is located at the origin (0,0) of the t-h plane.

II. ASSUMPTIONS

While the volume of our system is finite, its thermodynamics is completely determined by the free energy density $f_V(t,h)\equiv F_V(t,h)/V$, where F_V is the Helmholtz free energy of the system. In thermodynamics, only t and h control an equilibrium state of the system: no more parameters can come into play. As we approach the thermodynamic limit, our system develops a phase transition at (0,0), around which singular behaviors of various thermodynamic quantities (e.g., a divergence in the specific heat for a second order transition) are dictated by the singular part f_∞^S of the free energy density f_∞. For a finite system of volume V, we define the singular part of the free energy density as $f_V^S\equiv f_V-(f_\infty-f_\infty^S)$ [4], which converges to f_∞^S as $V\to\infty$. We also introduce the corresponding singular part of the free energy $F_V^S \equiv V f_V^S$. While f_V^S is an analytic function of t and h in some domain including (0,0), f_∞^S is not analytic at (0,0). Our central question is how the singular function f_∞^S emerges as the $V=\infty$ limit for the sequence of the analytic functions $\{f_V^S\}$. We will answer this question by deriving the finite size scaling ansatz starting from the following minimal set of basic assumptions on f_V^S, f_∞^S, and F_V^S:

(I) Assumption of analyticity: f_V^S and F_V^S are analytic in a domain around (0,0).

(II) Assumption of uniform convergence: the sequence $\{f_V^S\}$ converges uniformly to f_∞^S in the domain around (0,0), except at (0,0).

(III) Assumption of singularity: If the phase transition is of first order, then we assume that $f_\infty^S(t,0)=-u_\pm|t|$ and $f_\infty^S(0,h)=-m_\pm|h|$, where $u_+(t>0)$, $u_-(t<0)$, $m_+(h>0)$, and $m_-(h<0)$ are all positive constants. If the phase transition is of second order, then we assume that $f_\infty^S(t,0)=-a_\pm|t|^{2-\alpha}$ and $f_\infty^S(0,h)=-b_\pm|h|^{1/\delta+1}$, where $a_+(t>0$, $a_-(t<0)$, $b_+(h>0)$, $b_-(h<0)$, α, and δ are all positive constants. We can combine these two cases by assuming that $f_\infty^S(t,0)=-a_\pm|t|^{1/x_t}$ and $f_\infty^S(0,h)=-b_\pm|h|^{1/x_h}$, where for a first order transition $x_t=x_h=1$, $a_\pm=u_\pm$, and $b_\pm=m_\pm$, and for a second order transition $x_t=1/(2-\alpha)$ and $x_h=1/(1/\delta+1)$.

(IV) Assumption of renormalizability: $\lim_{V\to\infty} V(\partial f_V^S/\partial V)=0$.

(V) Without loss of generality, we assume that $F_V^S(0,0)=0$, because we can always redefine F_V by subtracting $F_V(0,0)$ from it so that $F_V(0,0)=f_V(0,0)=f_\infty(0,0)=0$ and $F_V^S(0,0)=f_V^S(0,0)=0$.

Assumption I follows from an assumption that f_V is analytic around (0,0). Assumption II is also based on an assumption that $\{f_V\}$ converges uniformly to f_∞. Both of these assumptions on f_V have been proven for some systems [5]. Assumption II also guarantees that F_V^S is extensive for large V: $F_{\lambda V}^S \approx \lambda F_V^S$ because $F_V^S \approx V f_\infty^S$.

Assumption III comes from the singular behaviors in thermodynamic quantities around (0,0) at the thermodynamic limit. For a first order transition, $f_\infty^S(t,0)=-u_\pm|t|$ leads to the singular part of the internal energy density $u^S \equiv -\partial f_\infty^S/\partial t$ that jumps from u_- to u_+ at $t=0$, while h is kept at zero. Similarly, $f_\infty^S(0,h)=-m_\pm|h|$ produces the singular part of the magnetization density $m^S \equiv -\partial f_\infty^S/\partial h$ that jumps from m_- to m_+ at $h=0$, while t is kept at zero. For a second order transition, $f_\infty^S(t,0)=-a_\pm|t|^{1/x_t}$ leads to the singular part of the specific heat $c^S \equiv -T^*\partial^2 f_\infty^S/\partial t^2$ that diverges as $c^S=c_\pm|t|^{-\alpha}$ (here $c_\pm=(2-\alpha)(1-\alpha)T^* a_\pm$ around $t=0$ while h is kept at zero. Similarly, $f_\infty^S(0,h)=-b_\pm|h|^{1/\delta+1}$ produces the singular part of the magnetization density that behaves as $m^S=m'_\pm|h|^{1/\delta}$ [here $m'_\pm=(1/\delta+1)b_\pm$] around $h=0$, while t is kept at zero.

Assumption IV is satisfied if f_V^S can be written, as is commonly assumed [6], as $f_V^S=f_\infty^S+\sigma_\infty^S/V^{1/d}+O(V^{-2/d})$, where σ_∞^S is the singular part of the $(d-1)$-dimensional surface free energy density. The reason for requiring this assumption as well as for calling it "assumption of renormalizability" will become clear in Sec. III.

III. THERMODYNAMIC RG TRANSFORMATIONS

We define a TRGT, which maps t and h of a system of volume V to t' and h' of another system of volume V', by demanding the singular part of the free energy to be preserved: $F_V^S(t,h)=F_{V'}^S(t',h')$. We will denote this TRGT as $(t',h')=R_{V'V}(t,h)$. We will show that we can always construct such TRGT's near the transition point (0,0), and that the TRGT is a device with which we can expose the way f_V^S approaches its thermodynamic limit f_∞^S.

The TRGT may appear somewhat similar to the conventional RG transformation (CRGT), which preserves the singular part of the free energy between a system with N degrees of freedom and the corresponding coarse-grained system with a smaller number of degrees of freedom, which can be obtained through a block spin transformation in real space or by integrating out the short wavelength degrees of freedom in momentum space. The main difference is that the TRGT maps only two macroscopic variables t and h to another pair t' and h', while the CRGT maps, by definition, a set of an infinite number of coupling constants in an effective Hamiltonian to another such set. In the CRGT, a coarse graining of the original degrees of freedom generates many additional coupling constants, whereas the TRGT does not involve such coarse graining. We will show that the TRGT's are one-to-one analytic transformations that form a group, whereas the CRGTs have, in general, no inverse transformation and therefore form a semigroup, and their analyticity is sometimes a delicate issue [7].

We will construct a TRGT in three steps. First we will construct a transformation between $(t,0)$ and $(t',0)$ by holding h at zero. We will then construct another transformation between $(0,h)$ and $(0,h')$ by holding t at zero. We can hold either t or h at zero, because they are external parameters that we can control independently of each other. Finally we will extend these two transformations to a domain around (0,0). According to assumption V, $F_V^S(0,0)=0=F_{V'}^S(0,0)$, and therefore the transition point (0,0) is a fixed point for any TRGT.

(i) $h=0$. We demand that $F_V^S(t,0)$ be preserved by the TRGT or be independent of the system volume V:

$$0=V\frac{dF_V^S(t,0)}{dV}=\left\{V\frac{\partial}{\partial V}+\beta_V(t)\frac{\partial}{\partial t}\right\}F_V^S(t,0) , \tag{1}$$

where

$$\beta_V(t)\equiv V\left[\frac{dt}{dV}\right]_{F_V^S(t,0)}=-\left[f_V^S+V\frac{\partial f_V^S}{\partial V}\right]\Big/\left[\frac{\partial f_V^S}{\partial t}\right] . \tag{2}$$

According to assumption II, $\lim_{V\to\infty}\partial f_V^S(t,0)/\partial t=df_\infty^S(t,0)/dt$, except at (0,0). Applying assumption IV in Eq. (2) and using assumption III, we find

$$\lim_{V\to\infty}\beta_V(t)\equiv\beta_\infty(t)=-1/\{d\ln[-f^S_\infty(t,0)]/dt\}$$
$$=-x_t t\ .$$

For large V, we can solve Eq. (1) or an RG equation for $h=0$ by the method of characteristics after replacing β_V with β_∞. By solving a characteristic equation, $dV/V=dt/\beta_\infty(t)$, we then obtain, for large V and V', $V^{x_t}t=V'^{x_t}t'$. The TRGT for $h=0$ is thus a rescaling transformation of t given by $(t',0)=R_{V'V}(t,0)=((V/V')^{x_t}t,0)$. We can then express the singular part of the free energy as $F^S_V(t,0)=\mathcal{A}(V^{x_t}t)$, where, according to assumption I, $\mathcal{A}$ is an analytic function. The existence of this TRGT crucially depends on the existence of the limit function β_∞, which is guaranteed by assumption IV. We therefore call assumption IV the assumption of "renormalizability."

The manner in which we have constructed the TRGT is analogous to the way we find, in lattice field theory (LFT), how the lattice constant a controls the bare coupling constant g as the continuum limit $a\to 0$ is approached. V and t in the TRGT correspond to $1/a$ and g in LFT, where we find g as a function of a by demanding the mass gap $m_a(g)\equiv(1/a)\tilde{m}(g)$, which corresponds to $F^S_V(t,0)$, to be independent of a.

(ii) $t=0$. As in (i), we demand that $F^S_V(0,h)$ be preserved by the TRGT:

$$0=V\{dF^S_V(0,h)/dV\}=\{V\partial/\partial V+\gamma_V(h)\partial/\partial h\}F^S_V(0,h)\ ,$$

where $\gamma_V\equiv V(dh/dV)_{F^S_V(0,h)}$, which, according to assumptions II, III, and IV, converges to $\gamma_\infty(h)=-1/\{d\ln[-f^S_\infty(0,h)]/dh\}=-x_h h$ as $V\to\infty$. For large V, we can solve this RG equation for $t=0$ by the method of characteristics after replacing γ_V with γ_∞. By solving a characteristic equation, $dV/V=dh/\gamma_\infty(h)$, we then obtain, for large V and V', $V^{x_h}h=V'^{x_h}h'$. The TRGT for $t=0$ is thus a rescaling transformation of h given by $(0,h')=R_{V'V}(0,h)=(0,(V/V')^{x_h}h)$. We can then express the singular part of the free energy as $F^S_V(0,h)=\mathcal{B}(V^{x_h}h)$, where, according to assumption I, $\mathcal{B}$ is an analytic function.

(iii) Extension. To extend our TRGT to a domain around (0,0), we assume that $|V^{x_t}t|\ll 1$ and $|V^{x_h}h|\ll 1$, so that we can expand F^S_V around (0,0) as $F^S_V(t,h)=\mathcal{A}'(0)(V^{x_t}t)+\mathcal{B}'(0)(V^{x_h}h)+O(t^2,h^2,th)$. If the system satisfies the time-reversal symmetry [i.e., $F_V(t,-h)=F_V(t,h)$], then we instead obtain

$$F^S_V(t,h)=\mathcal{A}'(0)(V^{x_t}t)+\{\mathcal{A}''(0)/2\}(V^{x_t}t)^2+\{\mathcal{B}''(0)/2\}(V^{x_h}h)^2+O(t^3,th^2)\ ,$$

because in this case $(\partial F^S_V/\partial h)_{(t,0)}=0$ and $(\partial^2F^S_V/\partial t\partial h)_0=0$. Therefore, for (t,h) close to (0,0), we find $F^S_V(t,h)\approx F^S_{V'}((V/V')^{x_t}t,(V/V')^{x_h}h)$, so that the TRGT is given by $(t',h')=R_{V'V}(t,h)=((V/V')^{x_t}t,(V/V')^{x_h}h)$, which is a one-to-one analytic rescaling transformation of t and h. We also expect that this TRGT can be extended to a region further away from (0,0) on the t-h plane for the following reasons. First, along both the t axis and the h axis, the TRGT can be extended over a longer interval centered around (0,0) because the functions $\mathcal{A}(V^{x_t}t)$ and $\mathcal{B}(V^{x_h}h)$ contain higher order terms in t and h, respectively. Second, because of the analyticity of F^S_V, if

$$F^S_V(t,h)=F^S_{V'}((V/V')^{x_t}t,(V/V')^{x_h}h)$$

holds in a small domain around (0,0), then it should also hold outside this domain. Assuming that the TRGT can be extended over a larger domain is analogous to an assumption made in the conventional RG theory that a linearized CRGT around a critical fixed point is diagonal in relevant variables t and h.

IV. FINITE SIZE SCALING FORM

Choose a large system of volume V_0 as a reference system with (t_0,h_0), and consider a TRGT from an arbitrary system of large volume V with (t,h) to this reference system: $t_0=(V/V_0)^{x_t}t$ and $h_0=(V/V_0)^{x_h}h$. According to the definition of the TRGT,

$$F^S_V(t,h)=F^S_{V_0}(t_0,h_0)$$
$$=F^S_{V_0}((V/V_0)^{x_t}t,(V/V_0)^{x_h}h)=\mathcal{F}(V^{x_t}t,V^{x_h}h)\ ,$$

where, according to assumption I, $\mathcal{F}(p,q)\equiv F^S_{V_0}(p/V_0^{x_t},q/V_0^{x_h})$ is an analytic function of arbitrary variables p and q. This result, $F^S_V=\mathcal{F}(V^{x_t}t,V^{x_h}h)$, is the finite size scaling form for F^S_V. Usually, the finite size scaling form is expressed in terms of the system's linear dimension L as $F^S_V=\mathcal{F}(L^{dx_t}t,L^{dx_h}h)$. Since $F^S_V(p/V^{x_t},q/V^{x_h})=F^S_{V_0}(p/V_0^{x_t},q/V_0^{x_h})$, $\mathcal{F}(p,q)$ does not depend on a choice of V_0 and is therefore uniquely determined. By taking a derivative of this finite size scaling form for F^S_V with respect to t or h, we can derive finite size scaling forms for the singular parts of the internal energy density, specific heat, magnetization density, and magnetic susceptibility [6].

(i) First order transitions. For this case, $x_t=x_h=1$, and we obtain $F_{V_S}=\mathcal{F}(L^dt,L^dh)$, which Fisher and Berker [3] postulated for these transitions.

(ii) Second order transitions. For this case, $x_t=1/(2-\alpha)$ and $x_h=1/(1/\delta+1)$, and $F^S_V=\mathcal{F}(L^{d/(2-\alpha)}t,L^{d/(1/\delta+1)}h)$, which is the ansatz proposed by Fisher and by Fisher and Barber [1].

V. SCALING LAWS AT THE THERMODYNAMIC LIMIT

Using the above finite size scaling form for F^S_V, we can show that, for f^s_v,

$$f_V^S(\lambda^{x_t}t,\lambda^{x_h}h)=(1/V)F_V^S(\lambda_{x_t}t,\lambda^{x_h}h)$$
$$=\{\lambda/(\lambda V)\}\mathcal{F}((\lambda V)^{x_t}t,(\lambda V)^{x_h}h)$$
$$=\lambda f_{\lambda V}^S(t,h)\ .$$

In the $V=\infty$ limit, f_∞^S is therefore a generalized homogeneous function: $f_\infty^S(\lambda^{x_t}t,\lambda^{x_h}h)=\lambda f_\infty^S(t,h)$, which leads to $f_\infty^S=|t|^{1/x_t}f_\infty^S(\text{sgn}(t),h/|t|^{x_h/x_t})$.

(i) First order transitions. For this case, $x_t=x_h=1$, and $f_\infty^S=|t|f_\infty^S(\text{sgn}(t),h/|t|)$.

(ii) Second order transitions. For this case, $x_t=1/(2-\alpha)$ and $x_h=1/(1/\delta+1)$, and $f_\infty^S=|t|^{2-\alpha}f_\infty^S(\text{sgn}(t),h/|t|^{\delta(2-\alpha)/(\delta+1)})$. By defining Δ as $\Delta\equiv\delta(2-\alpha)/(\delta+1)$, we obtain Widom's scaling hypothesis: $f_\infty^S=|t|^{2-\alpha}f_\infty^S(\text{sgn}(t),h/|t|^{\Delta})$, which directly leads, as is well known, to Rushbrook's scaling law $\alpha+2\beta+\gamma=2$, and Griffiths' law $\alpha+\beta(\delta+1)=2$.

VI. FINITE SIZE SCALING FUNCTIONS AND UNIVERSALITY

The finite size scaling function $\mathcal{F}(p,q)$ for $p\approx 0$ and $q\approx 0$ embodies the finite size rounding of the singularity of f_∞^S around the transition point, while asymptotic behaviors of the partial derivatives of $\mathcal{F}$ as $p\to\pm\infty$ or $q\to\pm\infty$ (e.g., $\lim_{p\to\pm\infty}|p|^{\alpha}\{\partial^2\mathcal{F}(p,0)/\partial p^2\}=-c_\pm/T^*$ for a second order transition) are determined by the singular behaviors of thermodynamic quantities at the thermodynamic limit (e.g., the divergence in the specific heat for a second order transition) [6]. Our thermodynamic approach says nothing about the universality [4] of the scaling function $\mathcal{F}$ among different systems. However, we can still define a universality class as a collection of systems whose scaling functions share the same asymptotic behaviors. Since these asymptotic behaviors are completely determined by the exponents x_t and x_h and amplitude ratios such as u_+/u_-, each universality class is also uniquely specified by them.

(i) First order transitions. Each universality class is specified by $x_t=x_h=1$, u_+/u_-, m_+/m_-, and u_+/m_+.

(ii) Second order transitions. Each universality class is specified by $x_t=1/(2-\alpha)$, $x_h=1/(1/\delta+1)$, c_+/c_-, m'_+/m'_-, m_+/m_-, χ_+/χ_-, $c_+/(T^*m'_+)$, $c_+/(T^*m_+)$, $c_+/(T^*\chi_+)$, m'_+/m_+, m'_+/χ_+, and m_+/χ_+, where $m^S(t<0,0)=m_\pm(-t)^{\beta}$ [here $m_+(h>0)$, and $m_-(h<0)$] and the singular part of the magnetic susceptibility $\chi^S(t,0)=\chi_\pm|t|^{-\gamma}$ [here $\chi_+(t>0),\chi_-(t<0)$].

VII. FINITE SIZE SCALING FORM FOR CORRELATION FUNCTIONS

To derive a finite size scaling form for a two-point correlation function which leads to Kadanoff's scaling hypothesis, we must generalize our approach by allowing the external field to be spatially varying on the macroscopic length scale. To this end, we first divide our system of volume V into a set of small but still macroscopic subsystems. We will set the number of subsystems to be a constant N_S. The number of spins in each subsystem is then proportional to V/N_S, which increases as V increases. We now introduce a slowly varying external field which is almost constant inside each subsystem so that we can label the external field as $h_i\equiv h(\vec{r}_i/V^{1/d})$, where $\vec{r}_i$ is the position of the center of the ith subsystem. We must also specify the strength u for the interfacial coupling between each pair of nearest-neighbor subsystems. We assume that the interactions among spins in our system are short ranged, so that we do not need to consider long-ranged couplings among the subsystems. Note that for a particular system, u takes a particular value and is therefore not a free parameter. It is, however, conceivable that we can vary u by inserting a buffer material into interfaces between subsystems. F_V^S then becomes $F_V^S(t,\{h_i\},u)$. We will construct a TRGT between two systems of volume V and V'. We assume these systems to have geometrically similar overall shapes, and we divide both systems into N_S subsystems in a similar way so that there exists a one-to-one correspondence between the subsystems of one of the systems and those of the other. We then construct a TRGT between h_i and h_i' by settering $t=0$, $h_j=0$ $(j\neq i)$, and $u=0$: $h_i'=((V/N_S)/(V'/N_S))^{x_h}h_i=(V/V')^{x_h}h_i$. To construct a TRGT between u and u', we set $t=0$ and $h_i=0$ for all i. Note that as the subsystem size grows, in order to preserve the interfacial free energy we must increase u. We thus postulate $u'=(V/V')^{x_u}u$, where $x_u<0$. We then obtain the finite size scaling form $F_V^S=\mathcal{F}(V^{x_t}t,\{V^{x_h}h_i\},V^{x_u}u)$. Noting that the number of spins in each subsystem is proportional to V/N_S, we find the two-point correlation function to be

$$G_V^S\equiv(V/N_S)^{-2}(\partial^2F_V^S/\partial h_i\partial h_j)$$
$$=V^{2(x_h-1)}\mathcal{G}(r/V^{1/d},V^{x_t}t,\{V^{x_h}h_k\},V^{x_u}u)\ ,$$

where $r=|\vec{r}_i-\vec{r}_j|$. This leads to

$$G_{\lambda V}^S(r,t,\{h_k\},u)=\lambda^{2(x_h-1)}G_V^S(r/\lambda^{1/d},\lambda^{x_t}t,\{\lambda^{x_h}h_k\},\lambda^{x_u}u)\ .$$

By setting all $h_i=0$ after taking the $V=\infty$ limit, we obtain

$$G^S(r,t,u)\equiv G_\infty^S(r,t,\{0\},u)=\lambda^{2(x_h-1)}G^S(r/\lambda^{1/d},\lambda^{x_t}t,\lambda^{x_u}u)$$

which, for large r, leads to

$$G^S(r,t,u)=r^{2d(x_h-1)}G^S(1,r^{dx_t}t,r^{dx_u}u)$$
$$\approx r^{2d(x_h-1)}G^S(1,r^{dx_t}t,0)\ ,$$

because $x_u<0$ [8].

(i) First order transitions. For this case, $x_t=x_h=1$, and $G^S(r,t,u)=G^S(1,r^dt,0)$. For $t=0$, $G^S(r,0,u)=G^S(1,0,0)$, or $G^S(r,0,u)$ is constant. This is consistent with our expectation for first order transitions, where the correlation length remains finite so that the correlation function vanishes for large r.

(ii) Second order transitions. For this case, $x_t=1/(2-\alpha)$ and $x_h=1/(1/\delta+1)$, and we find

$$G^S(r,t,u)=r^{-2d/(\delta+1)}G^S(1,\text{sgn}(t)(r|t|^{(2-\alpha)/d})^{d/(2-\alpha)},0)\ ,$$

which is Kadanoff's scaling hypothesis: $G^S = r^{-(d-2+\eta)}\mathcal{H}(rt^{\nu})$. By comparing these expressions for G^S, we can derive Josephson's hyperscaling law $2-\alpha=d\nu$ [8], and Fisher's scaling law $\gamma=\nu(2-\eta)$.

VIII. SUMMARY

For a system with a first or second order phase transition, we have constructed a thermodynamic renormalization group transformation (TRGT) that maps the reduced temperature t and the external field h for a system of volume V to t' and h' for another system of volume V'. Our derivation has been purely thermodynamic, and based on a minimal set of assumptions on the singular parts of the free energy and of the free energy density both near and at the thermodynamic limit. Our TRGT is a one-to-one analytic rescaling transformation of macroscopic variables t and h. The TRGT's allow us to derive the finite size scaling form for the singular part of the free energy, which leads to the scaling laws at the thermodynamic limit. We have thus shown that the finite size scaling of first and second order phase transitions and the scaling laws at the thermodynamic limit are all truly thermodynamic results. The thermodynamic renormalization group approach presented in this paper therefore provides a unified thermodynamic framework for phase transitions.

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[8] For some systems (e.g., the Ising model in $d>4$ dimensions), the $u=0$ limit may be singular and $\lim_{y\to 0}G^S(1,x,y)=y^{\mu}\overline{G}(xy^{\sigma})$. For these systems, Josephson's hyperscaling law is violated. For the Ising model in $d>4$ dimensions, $\mu=\sigma=1$, $dx_u=-(d-4)/2$, $\alpha=0$, and $\delta=3$ give the correlation function with $\eta=0$ and $\nu=\frac{1}{2}$.